

On the Rank Number of Grid Graphs

Sitan Chen

Harvard College, Cambridge, MA 02138

Abstract

A vertex k -ranking is a labeling of the vertices of a graph with integers from 1 to k so any path connecting two vertices with the same label will pass through a vertex with a greater label. The *rank number* of a graph is defined to be the minimum possible k for which a k -ranking exists for that graph. For $m \times n$ grid graphs, the rank number has been found only for $m \leq 3$. In this paper, we determine its exact value when $m = 4$ and improve the upper bound for the rank number of general grids. Furthermore, we improve lower bounds on the rank numbers for square and triangle grid graphs from logarithmic to linear. These new lower bounds are key to characterizing the rank number for general grids, a major open problem in graph labeling. The novel methods we use to obtain these results can be used to study minimal separators of graphs, which have applications in optimizing VLSI circuit design and a variety of parallel algorithms, including parallel query processing, searching for corruptions in partially ordered data structures, and parallel scheduling of the assembly of multi-part products in manufacturing systems.

1. Introduction

Given a graph G , a *graph labeling* is a function that takes the vertices $V(G)$ in a graph G to a subset of the integers, subject to certain constraints. One notable graph labeling, a k -coloring, labels a graph with k colors so no two adjacent vertices share the same color [6].

The very large-scale integration (VLSI) circuit layout motivated the study of k -ranking, a generalization of k -coloring. A VLSI circuit consists of a large number of transistors and wires contained within a multi-layer chip. If we treat transistors as vertices and wires as edges in a graph, then many graph properties are related to circuit features [1, 3]. In particular, k -ranking was first studied for its connections to finding minimal separators of graphs [8, 11]. Finding minimal separators is key to minimizing VLSI layout area, which is directly related to hardware expense [10].

In this paper, we first introduce some definitions and previous results in Section 2. Then in Section 3, we determine the closed form for the rank number of $G_{4,n}$, and in section 4, we improve previous upper bounds for $\chi_r(G_{m,n})$. In [2], Alpert called for a strong lower bound for $G_{n,n}$, which is key to completely determining $\chi_r(G_{m,n})$, so in Section 5, we present a new lower bound for $G_{n,n}$, improved from logarithmic to linear. As a corollary, this improves the lower bound for the rank number of triangle grids tri_n (see Figure 1g) from logarithmic to linear.

The results on $r(4, n)$ and $r(m, n)$ and the new methods used to obtain them offer insight on how to approach completely determining the rank number of grid graphs given a suitable lower bound on the rank number of square grid graphs, one that we provide. Combined, the results of this research provide substantial groundwork for solving the open problem of determining $r(m, n)$.

Furthermore, the novel methods we introduce to study minimal separators in obtaining an improved lower bound on $G_{n,n}$ are key not only to optimizing VLSI circuit design, but also to optimizing many parallel algorithms, including parallel scheduling of multi-part product assembly in manufacturing systems, searching for corruptions in partially ordered data structures, parallel query processing, and Cholesky factorizing matrices in parallel [5].

2. Fundamentals

Gallian [7] gives the following definition.

Definition. A labeling function $f : V(G) \rightarrow \{1, \dots, k\}$ is a k -ranking of a graph G if each path between two vertices of the same value passes through a vertex with a larger value. The rank number $\chi_r(G)$ is the smallest k for which a k -ranking of G exists. Furthermore, a k -ranking is minimal if no label can be replaced with a smaller value and still satisfy the conditions for a ranking.

Note this is a generalization of k -coloring, which only involves paths of length 1, to paths of arbitrary length. We can make several self-evident observations.

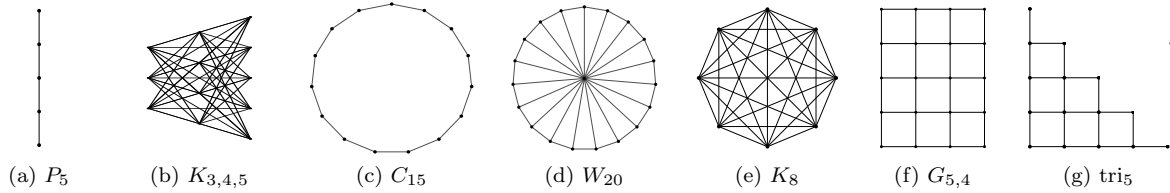
Lemma 1. For a connected graph, the largest value k used in its k -ranking is unique.

Lemma 2. If H is a subgraph of G , then $\chi_r(H) \leq \chi_r(G)$.

We combine lemmas 1 and 2 to get the following lemma that the rank number monotonically increases with the addition of vertices [9].

Lemma 3. For any vertex $v \in G$, $\chi_r(G) - 1 \leq \chi_r(G - v) \leq \chi_r(G)$.

According to Gallian [7], the rank number of a path graph is known to be $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1 = 1 + \chi_r(P_{\lceil \frac{n-1}{2} \rceil})$. Likewise complete t -partite graphs K_{s_1, s_2, \dots, s_t} have a rank number of $n+1 - |s_k|$, where $|s_k|$ is the number of vertices in the largest partite set s_k . For cycles, the rank number is $\chi_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2$. For wheels, the rank number is $\chi_r(W_n) = \lfloor \log_2(n-3) \rfloor + 3$. For complete graphs, the rank number is $\chi_r(K_n) = n$.



Chang et al. [4] showed that $r(2, n) = 2 + r(2, \lfloor \frac{n-2}{2} \rfloor)$; Alpert [2] showed that

$$r(3, n) = \begin{cases} 4 + r(3, \lfloor \frac{n-3}{2} \rfloor) & \text{for } n = 15 \cdot 4^k + 7 \cdot \frac{4^k - 1}{3} + \frac{3 \pm 1}{2} \\ 3 + r(3, \lfloor \frac{n-3}{2} \rfloor) & \text{for } n \neq 15 \cdot 4^k + 7 \cdot \frac{4^k - 1}{3} + \frac{3 \pm 1}{2} \end{cases}$$

Finding the rank number of a grid graph $G_{m,n}$, however, is an open problem for $m \geq 4$.

3. Rank Number of $G_{4,n}$

For simplicity, we denote the rank number of $G_{m,n}$ by $r(m, n)$. Let $b(n) = 2b_2 + b_3$, where b_i is the i th most significant bit of n .

Theorem 1. For $n > 8$ if $n = 2^k + 2^{k-2} - 2$ or $2^k + 2^{k-2} - 1$, then $r(n) = 4k - 2$. Otherwise,

$$r(4, n) = 4 \lfloor \log_2(n+1) \rfloor - 3 + b(n+1).$$

Alternatively, if $I = [2^k - 1, 2^k] \cup [2^k + 2^{k-1} - 1, 2^k + 2^{k-1}] \cup [2^k + 2^{k-1} + 2^{k-2} - 1, 2^k + 2^{k-1} + 2^{k-2}]$, then

$$r(4, n) = \begin{cases} 5 + r\left(4, \left\lfloor \frac{n-4}{2} \right\rfloor\right) & \text{if } n \in I \\ 4 + r\left(4, \left\lfloor \frac{n-4}{2} \right\rfloor\right) & \text{if } n \notin I \end{cases}$$

Before we prove this, we define some terms. Let α be the largest value that is used more than once in a given minimal ranking f of $G_{4,n}$ and T_α the set of vertices labeled with values greater than α . Let the *cut set* $C(f)$ be a minimal subset of vertices in T_α so that the vertices labeled α are in different connected components of $G \setminus C(f)$, and let $|C(f)|$ be the number of vertices in $C(f)$.

We first find tight lower and upper bounds for the rank number of $4 \times n$ grids.

3.1. Lower Bound

Lemma 4. For $n > 5$,

$$r(4, n) \geq 4 + r\left(4, \left\lceil \frac{n-4}{2} \right\rceil\right).$$

Proof. For $G_{4,n}$, with $n > 1$, clearly $|C(f)| > 1$. If $|C(f)| = 2$, then the cut set must be the two neighbors of a corner, and α is the value of the corner. Because the ranking is minimal, $\alpha = 1$, but if $n > 2$, then $\alpha > 1$. Similarly, if $|C(f)| = 3$, then it is easy to check that any configuration of cut set vertices will split the graph into two connected components, one of which must have a rank number of at most 2. Again, because the ranking is minimal, $\alpha \leq 2$, but if $n > 3$ then $\alpha > 2$. Thus, for $n > 3$, we have that $|C(f)| \geq 4$. In the cut set, any four vertices will occupy at most four consecutive columns, and $r(4, n) \geq 4 + r\left(4, \left\lceil \frac{n-4}{2} \right\rceil\right)$, because the complement of those four vertices will contain $G_{4, \lceil \frac{n-4}{2} \rceil}$. \square

3.2. Upper Bound

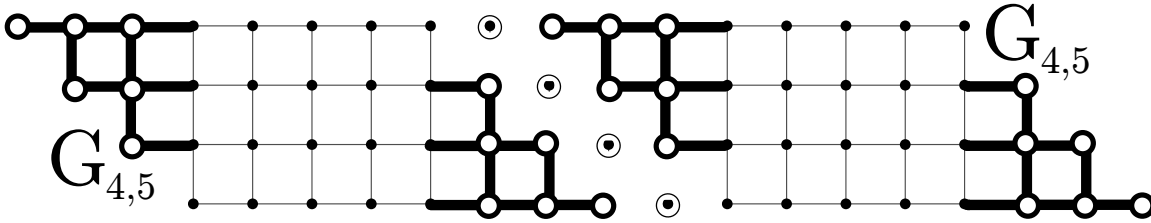


Figure 1: Merging two 4×5 grids with sticky ends (bolded) to get a 4×14 grid with sticky ends.

We next find the upper bound. Define a *sticky end* to be the staircase-like configuration attached to the end of a grid, each of which is bolded in Figure 1.

Lemma 5. If $G_{4,n}$ with a sticky end on both sides can be labeled with λ colors, then

$$r(4, 2n+4) \leq \lambda + 4.$$

Proof. Place two such $G_{4,n}$ graphs with sticky ends, both colored in the same way, together and let the cut set be four vertices between these two graphs, colored with $\lambda + 1$, $\lambda + 2$, $\lambda + 3$, and $\lambda + 4$. This gives a $(\lambda + 4)$ -ranking of $G_{4,2n+4}$ with two sticky ends. Because $G_{4,2n+4}$ with no sticky ends is its subgraph, $r(4, 2n+4) \leq \lambda + 4$. \square

Figure 1 shows an example for constructing $G_{4,14}$ with two sticky ends out of two copies of $G_{4,5}$ with two sticky ends.

In [2], Alpert showed that $r(4, 3) = 6$. We can show by exhaustive search and construction in Figure 2 that $r(4, 4) = 7$, $r(4, 5) = 8$, $r(4, 6) = 8$, $r(4, 7) = 9$, and $r(4, 8) = 10$. Define $s(4, n)$ to be the rank number of $G_{4,n}$ with two sticky ends. We can also show by construction in Figure 3 that $s(4, 1) \leq 5$, $s(4, 2) \leq 6$, $s(4, 3) \leq 7$, $s(4, 4) \leq 8$.

We rewrite the lower bound as $r(4, 2n+3) \geq 4 + r(4, n)$ and the upper bound as $r(4, 2n+4) \leq s(2n+4) \leq 4 + s(n)$ and combine these bounds so that for each n , $r(4, n)$ is bounded between two consecutive integers. We can condense these combined bounds to get that for $k \geq 3$, where $k = \lfloor \log_2(n+3) \rfloor$,

$$4k - 4 + i \leq r(4, n) \leq 4k - 3 + i \quad \text{for } n \in [B_{k,i}, B_{k,i+1}), \quad (1)$$

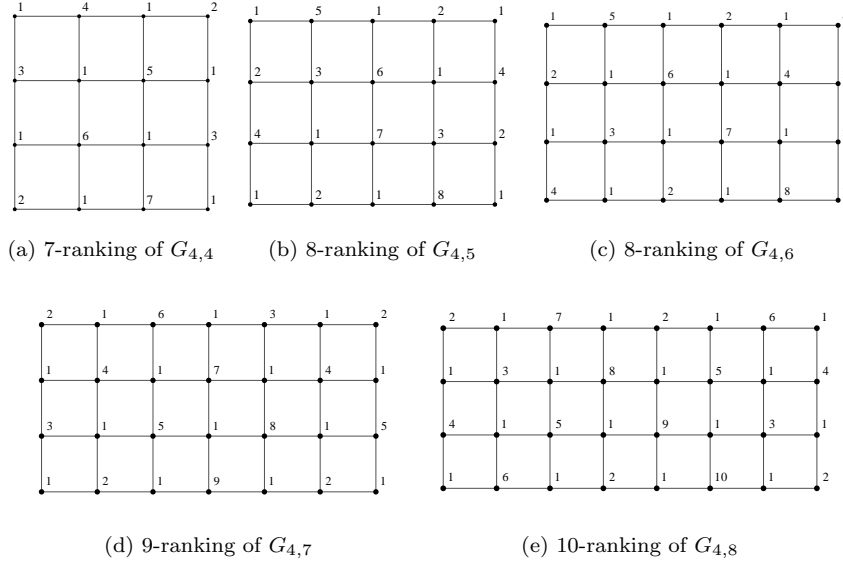


Figure 2: $r(n)$ -ranking of $G_{4,n}$

where $0 \leq i \leq 3$ and $B_{k,0} = 2^k - 3$, $B_{k,1} = 2^k + 2^{k-2} - 3$, $B_{k,2} = 2^k + 2^{k-1} - 3$, $B_{k,3} = 2^k + 2^{k-1} + 2^{k-2} - 3$, and $B_{k,4} = 2^{k+1} - 3$.

3.3. Completely Characterizing $r(4, n)$

We restrict to cases where $n > 8$ and

$$|C(f)| = 4.$$

Note that the latter preserves generality for the arguments following this lemma because given a cut set of four vertices and the corresponding widest possible induced subgrid, an addition of k extra vertices to the cut set will occupy at most k vertices of that subgrid. Thus, the rank number of the subgrid with these vertices deleted is at most k less than the original subgrid's rank number, while the addition of k vertices to the cut set increases the rank number by k . This, combined with the result in section 3.1 that $|C(f)| \geq 4$, lets us restrict our attention to grids of exactly four cut set vertices for the rest of Section 3.

In each interval $[B_{k,i}, B_{k,i+1})$, we aim to find the closed form of n such that $r(4, n+1) > r(4, n)$.

We first prove two lemmas. The first, the Merging Lemma, will provide a construction that we will show for each integer $0 \leq i \leq 3$ guarantees the existence of a $(4k - 4 + i)$ -ranking for all G_{4,n_1} , where $n_1 \in [B_{k,i}, M]$ for some integer $M \leq B_{k,i+1}$. The second, the Corner Lemma, will be used to show there only exists a $(4k - 3 + i)$ -ranking for all G_{4,n_2} , where $n_2 \in (M, B_{k,i+1})$.

Merging Lemma. *If $G_{4,n}$ with a sticky end has a λ -ranking and $G_{4,n-1}$ with two sticky ends has a λ -ranking, then $G_{4,2n+3}$ with a sticky end has a $(\lambda + 4)$ -ranking and $G_{4,4n+10}$ has a $(\lambda + 8)$ -ranking.*

Proof. If G_{4n+10} is bisected with a cut set of 4 high vertices descending diagonally from left to right in the middle, then the induced subgraphs are two copies of $G_{4,2n+3}$ each with one sticky end. Each induced subgraph is further bisected with a cut set of 4 high vertices parallel to the sticky end to get a $G_{4,n}$ subgraph with a sticky end and a $G_{4,n-1}$ subgraph with two sticky ends. Both have λ -rankings, so $G_{4,2n+3}$ with a sticky end has a $(\lambda + 4)$ -ranking and $G_{4,4n+10}$ has a $(\lambda + 8)$ -ranking. \square

This division of $G_{4,4n+10}$ is shown for $n = 5$ in Figure 4.

Next, define a *grid graph G missing a corner* to be $G \setminus v$, where v is a corner vertex.

Corner Lemma. *One of the two connected components induced from removing the cut set from $G_{4,n}$ contains the subgraph $G_{4,ceil \frac{n-2}{2}}$ missing one corner.*

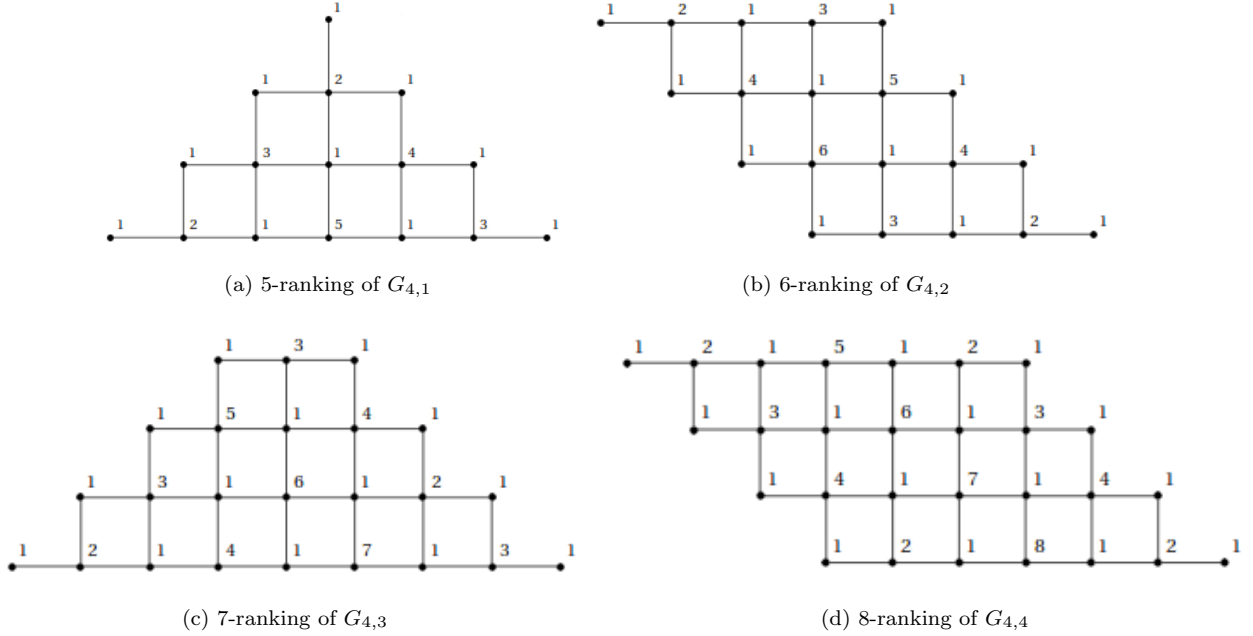


Figure 3: $r(n)$ -ranking of $G_{4,n}$ with two sticky ends

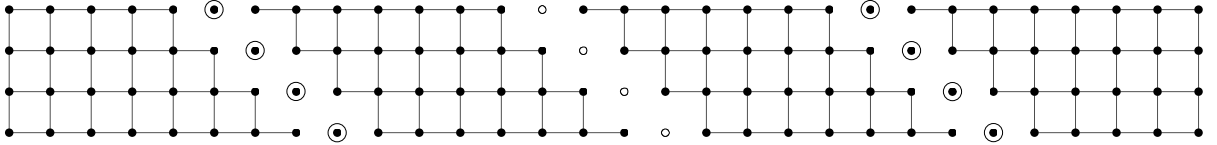


Figure 4: $G_{4,2^5-2}$ split by 4 high vertices (white) into two copies of $G_{4,2^4-3}$, which are each split by 4 high vertices (circled) into G_{2^3-3} with a sticky end and G_{2^3-4} with two sticky ends

Proof. It is easy to see that this is true if the cut set occupies two or fewer consecutive columns. If the cut set occupies three columns, then at least one of the vertices in the cut set on the top or bottom row will not share a column with the rest of the cut set. If we remove the other three points in two columns, we induce a $G_{4,\lceil \frac{n-2}{2} \rceil}$ subgraph, and by removing the remaining point in the cut set, we remove its corner.

If the cut set occupies four columns, then we can similarly remove the middle two columns to induce a $G_{4,\lfloor \frac{n-2}{2} \rfloor}$ subgraph, and by removing either of the two remaining points, we remove its corner. \square

Corollary. For $n \geq 5$, if the rank number of $G_{4,\lceil \frac{n-2}{2} \rceil}$ missing two corners in different columns is λ , then the rank number of $G_{4,n}$ missing two corners in different columns is at least $\lambda + 4$.

Proof. The Corner Lemma applied to $G_{4,n}$ missing two corners induces one connected component containing a subgrid one of whose missing vertices is a cut set vertex of $G_{4,n}$. The other missing vertex is one of the two missing vertices of $G_{4,n}$. This gives the desired lower bound. \square

This corollary illustrated in Figure 5 for $n = 7$.

Now we are ready to prove our main theorem. We proceed by casework for the intervals in which n resides.

Case 1. If $B_0 \leq n < B_1$, then $r(4, n + 1) \geq r(4, n)$ only if $n = B_0 + 1 = 2^k - 2$.

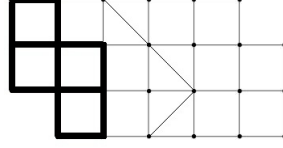


Figure 5: Removing any cut set of $G_{4,7}$ missing two corners in different columns induces a subgraph of $G_{4,3}$ minus two corners in different columns

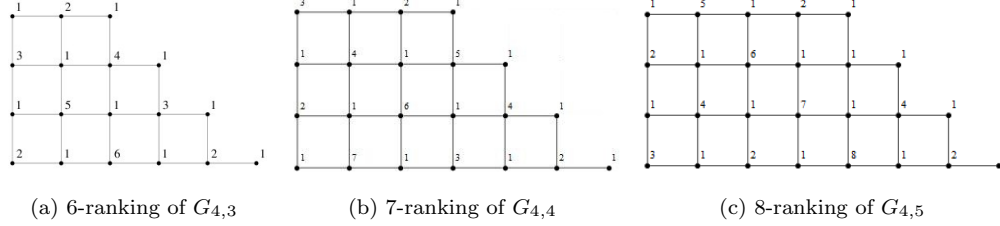


Figure 6: $r(n)$ -ranking of $G_{4,n}$ with one sticky end

Proof. We show an 8-ranking of $G_{4,5}$ with one sticky end in Figure 6c. Then by iteration of the Merging Lemma, there exists a $4k - 4$ ranking of $G_{4,2k-2}$. However, for $G_{4,2k-1}$, by the Corner Lemma, any cut set will induce $G_{4,2k-1-1}$ minus a corner. Because $G_{4,3}$ missing two corners has a rank number of 5, as shown in Figure 7a, by the corollary of the Corner Lemma, $G_{4,2k-1-1}$ missing two corners has a rank number of at least $4k - 7$, so $r(2^k - 1) \geq 4k - 3$. \square

Case 2. If $B_1 \leq n < B_2$, then $r(4, n) = 4k - 2$.

Proof. For $G_{4,2^k+2^{k-2}-3}$ when $k > 4$, by the Corner Lemma, any cut set will induce a subgraph $G_{4,2^{k-1}+2^{k-3}-2}$ minus a corner.

To show that $G_{4,18}$ minus a corner has no 13-ranking, consider the rightmost 4×8 subgrid that is missing a corner, which we will call $G_{4,8}^\dagger$. If G^{dagger} contains a cut set vertex of $G_{4,18}$, then the leftmost 4×8 subgrid will contain at most one cut set vertex of $G_{4,18}$. Otherwise, $G_{4,8}^\dagger$ contains no such cut set vertex, so we want to prove that a 4×8 grid minus a corner has no 9-ranking. Note that any cut set of $G_{4,8}^\dagger$ must induce a tri_5 subgraph missing the two outermost vertices of its first column and row, for which no 5-ranking exists. Hence, $r(4, 18) = 14$.

In general, $G_{4,2^k+2^{k-2}-3}$ is a subgraph of $G_{4,2^k+2^{k-2}-2}$ missing a corner, and removing any cut set from the former induces a subgraph of $G_{4,2^{k-1}+2^{k-3}-2}$ missing a corner, which by the inductive hypothesis has a rank number of $4k - 6$, so the rank number of $G_{4,2^k+2^{k-2}-2}$ missing a corner is at least $4k - 2$. Thus, $r(2^k + 2^{k-1} - 3) = 4k - 2$. \square

Case 3. If $B_2 \leq n < B_3$, then $r(4, n + 1) > r(4, n)$ only if $n = B_2 + 1 = 2^k + 2^{k-1} - 2$.

Proof. We show a 6-ranking of $G_{4,3}$ with one sticky end in Figure 6a. Then by iteration of the Merging Lemma, there exists a $(4k - 2)$ -ranking of $G_{4,2^k+2^{k-1}-2}$.

However, for $G_{4,2^k+2^{k-1}-1}$, by the Corner Lemma, any cut set will induce $G_{4,2^{k-1}+2^{k-2}-1}$ minus a corner. While $G_{4,5}$ minus two corners in the same row has a rank number of 6, $G_{4,5}$ minus one corner has a rank number of 7, shown in Figure 7b.

By (3.2), $r(4, 11) \leq 11$, so the rank number of $G_{4,11}$ minus two corners is at most 11. To prove that no 10-ranking exists, first consider the case in which the missing corners are in the same row but different columns. Assume without loss of generality that the topmost cut set vertex lies on or to the left of the middle column of $G_{4,11}$ minus two corners. Then the cut set will always induce either a $G_{4,5}$ grid minus two corners on the same row with a vertex appended to the remaining row, or a 4×5 grid minus two corners in different rows and columns. If the former subgrid is induced, any 3-vertex cut set of that subgraph will induce either

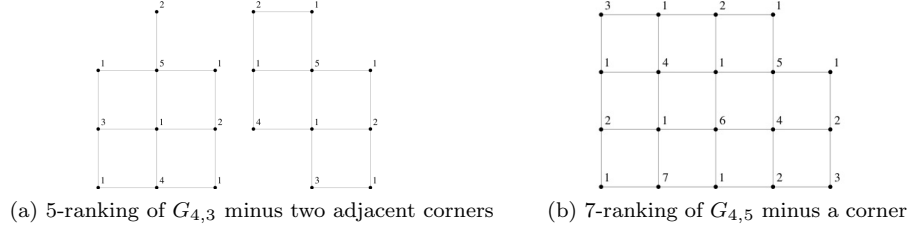


Figure 7: $r(n)$ -ranking of $G_{4,n}$ missing one or more corners

a path with 8 vertices or a 2×2 grid with a vertex appended to each side of one row or column, neither of which has a 3-ranking, while any 4-vertex cut set will induce a 2×2 grid, which has no 4-ranking. On the other hand, if the latter subgrid is induced, any 3-vertex cut set of that subgraph will induce a 2×3 grid, which has no 3-ranking, while any 4-vertex cut will induce a 2×2 grid, which has no 2-ranking.

Next, consider the case in which the missing corners are in different rows and columns. If the topmost cut set vertex lies to the right of the middle column of $G_{4,11}$ minus two corners, then the cut set will always induce a 4×5 grid missing two corners in different rows and columns, which we have shown has no 6-ranking. On the other hand, if the topmost cut set vertex lies on or to the left of the middle column, then the cut set will always induce a 4×5 grid minus two corners on the same row with a vertex appended to the remaining row, which has also been shown to have no 6-ranking.

Thus, the rank number of $G_{4,11}$ minus two corners is 11, so by the corollary of the Corner Lemma, $G_{4,2^k-1+2^{k-2}-1}$ missing two corners has a rank number of at least $4k - 5$, so $r(2^k + 2^{k-1} - 1) \geq 4k - 1$. \square

Case 4. If $B_3 \leq n < B_4$, then $r(4, n + 1) > r(4, n)$ only if $n = B_3 + 1 = 2^k + 2^{k-1} + 2^{k-2} - 2$.

Proof. We show a 7-ranking of $G_{4,4}$ with one sticky end in Figure 6b. Then by iteration of the Merging Lemma, there exists a $4k - 1$ ranking of $G_{4,2^k+2^{k-1}+2^{k-2}-2}$.

However, for $G_{4,2^k+2^{k-1}+2^{k-2}-1}$, by the Corner Lemma, any cut set will induce a subgraph $G_{4,2^{k-1}+2^{k-2}+2^{k-3}-1}$ minus a corner. While $G_{4,6}$ minus corners in different rows and columns has a rank number of 7, $G_{4,6}$ minus one corner has a rank number of 8.

By (3.2), $r(4, 13) \leq 12$, so the rank number of $G_{4,13}$ minus two corners is at most 12. The proof that no 11-ranking exists is roughly the same as the proof in case 3 that no 10-ranking exists for $G_{4,11}$ minus two corners. First, consider the case in which the missing corners are in the same row but different columns. Assume without loss of generality that the topmost cut set vertex lies on or to the left of the middle column of $G_{4,13}$ minus two corners. Then the cut set will always induce either a 4×6 grid missing two corners in different rows and columns with a vertex appended to one of these rows, or a 4×6 grid missing two corners on the same row. If the former subgraph is induced, any 3-vertex cut set of that subgraph will induce a 3×3 grid, which has no 4-ranking, while any 4-vertex cut set will induce a 2×3 grid or a path of 8 vertices, neither of which has a 3-ranking. On the other hand, if the latter subgrid is induced, any 3-vertex cut set of that subgraph will induce two 2×3 grids joined by a 2-vertex path subgraph, which together have no 4-ranking, while any 4-vertex cut will induce a 2×3 grid, which has no 3-ranking.

Next, consider the case in which the missing corners are in different rows and columns. If the topmost cut set vertex lies on or to the left of the middle column of $G_{4,13}$ minus two corners, then the cut set will always induce a 4×6 grid missing two corners in the same row, which we have shown has no 8-ranking. On the other hand, if the topmost cut set vertex lies to the right of the middle column, then the cut set will always induce a 4×6 grid missing two corners in different rows and columns with a vertex appended to one of these rows, which we have also shown to have no 8-ranking.

By the corollary of the Corner Lemma, $G_{4,2^k-1+2^{k-2}+2^{k-3}-1}$ missing two corners has a rank number of at least $4k - 4$, so $r(2^k + 2^{k-1} + 2^{k-2} - 1) \geq 4k$. \square

We have completely determined the rank number for $4 \times n$ grid graphs.

4. Upper Bound for General Grids

In [2], Alpert showed that $r(m, n) \leq m + r(m, \lceil \frac{n-1}{2} \rceil)$ by constructing a ranking with m high vertices in the middle column and $r(m, \lceil \frac{n-1}{2} \rceil)$ smaller colors ranking the larger connected component [2]. Here we present an improved upper bound.

For simplicity, let $\chi_r(\text{tri}_n) = \text{tri}(n)$. The subgraphs induced by removing a diagonal cut set of m vertices from the middle of $G_{m,n}$ are $G_{m, \lfloor \frac{n-m}{2} \rfloor}$ with one sticky end and $G_{m, \lceil \frac{n-m}{2} \rceil}$ with one sticky end, respectively, as shown in Figure 8. The rank number of the latter is greater than that of the former. Furthermore, $G_{m, \lceil \frac{n-m}{2} \rceil}$ with one sticky end can be broken up into $G_{m, \lceil \frac{n-m}{2} \rceil - 1}$ and tri_m , so that $G_{m, \lceil \frac{n-m}{2} \rceil - 1}$ is colored with the integers from 1 to $r(m, \lceil \frac{n-m}{2} \rceil - 1)$, inclusive, while tri_m is colored with the integers from $r(m, \lceil \frac{n-m}{2} \rceil - 1) + 1$ to $r(m, \lceil \frac{n-m}{2} \rceil - 1) + \text{tri}(m)$, inclusive.

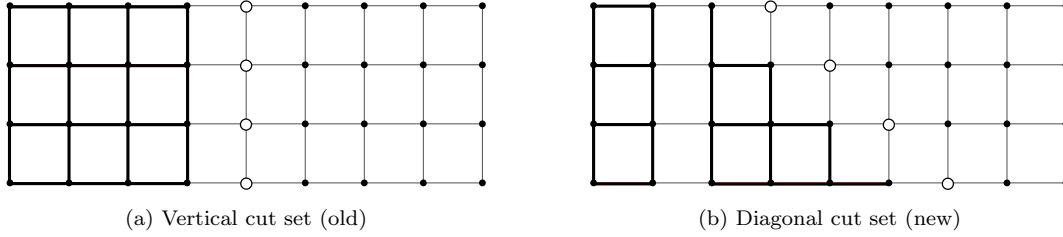


Figure 8: Comparison of cutting methods

In [2], Alpert showed that $\text{tri}(n) \leq n - 1 + \text{tri}(\lceil \frac{n-2}{2} \rceil)$. For $n \geq 3$, we can iterate this to get $\text{tri}(n) \leq 2n - 2 \lfloor \log_2(n+1) \rfloor + 1$. Using this, we get a new upper bound of

$$r(m, n) \leq 3m - 2 \log_2(m+1) + 1 + r\left(m, \left\lceil \frac{n-m}{2} \right\rceil - 1\right).$$

For $n \leq N$, where $N = O\left(m^{\frac{3}{2}}\right)$, our bound is tighter.

5. Lower Bound for Square Grid Graphs

We improve the known lower bound of $r(m, m)$ from logarithmic to linear.

Theorem 2. For $m \geq 5$, $r(m, m) \geq m + r\left(\left\lceil \frac{2m}{5} \right\rceil - 1, \left\lceil \frac{2m}{5} \right\rceil - 1\right)$.

First note that for $|C(f)| < m$, one of the subgraphs induced by removing the cut set will contain $G_{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil}$. Without loss of generality, let the top and bottom rows of the square grid both contain at least one cut set vertex (in rough terms, the cut set “goes from top to bottom”). Furthermore, let the number of consecutive columns that the cut set occupies in S_m be k . Clearly our lower bound holds for $k = 1$, so now, we only consider cut sets at least two sets wide.

We first define some terms, examples of which are illustrated in Figure 9a. Define two cut set vertices to be *cut-neighbors* if their column and row indices each differ by at most 1. A *cutpath* between two points v_0 and v_n is the sequence of cut set vertices $[v_0, v_1, \dots, v_n]$, where v_i and v_{i+1} are cut-neighbors for all $0 \leq i < n$ and a *sub-cutpath* to be any subsequence of the cutpath. Because some rows will contain multiple cut set vertices, there is no bijection between a cut set and its column sequence. Instead, call the bottommost row of the cut set row 0; we will denote a vertex in row r and column c as vertex (r, c) and a sub-cutpath as $[(r_0, c_0), (r_1, c_1), \dots, (r_n, c_n)]$.

Note that for $|C(f)| = m$, there is a bijection between the sequence of vertices in a cutpath and the sequence of column indices of those respective vertices. Denote this *column sequence* by $[c_0, c_1, \dots, c_n]$, where c_i is the index of the column that v_i occupies.

We will let the index of the leftmost column be 0 and that of the rightmost column be $k - 1$. Then define a *left-diagonal cutpath* (*right-diagonal cutpath*) to be a cutpath with column sequence $[0, 1, \dots, k - 1, k - 2, \dots, 1, 0]$ ($[k - 1, k - 2, \dots, 0, 1, \dots, k - 2, k - 1]$).

Define a *left-deviation* (*right-deviation*) to be a subsequence of the form $[c_i, c_{i+1}]$ in the cutpath's column sequence, where v_{i+1} is in a row higher (lower) than that of v_i and $c_{i+1} - c_i \leq 0$ ($c_{i+1} - c_i \geq 0$) and the closest vertices preceding and following it in the cutpath that lie on columns 0 and $k - 1$ lie, respectively, on column 0 (column $k - 1$) and column $k - 1$ (column 0).

Define a *deformation of v* as the shifting of v and all vertices following v in the cutpath up one unit, as long as v is still a cut-neighbor of the vertex preceding it in the cutpath.

For a rectangular subgrid $G_{x,y}$ with its leftmost column on the leftmost column of the original $G_{m,m}$ grid, define an *extension* to be its supergraph $G_{x,y+1}$ minus the rows containing vertices on the cutpath. The i th extension is defined to be the extension of the $(i - 1)$ th extension. Note that if a given subgrid in S must exist, then its extension must also exist. An example is shown for $G_{10,10}$ and $k = 7$ in Figure 9a. Finally, define the k th subgrid extension to be the subgrid $G_{x,y}$ in $S_{m,k}$ so that $(x, y) = 2k - 3 - 2i, \lceil \frac{m-k}{2} \rceil + i$.

To prove our lower bound for $r(m, m)$, we will essentially show that for any given k , the cut set that gives a ranking of fewest labels is a diagonal cutpath (obviously, orientation is irrelevant). We will first deal with cut sets of m vertices, for which we will show in Lemma 6 that for a given k , a certain family of subgrids is guaranteed to exist for any cut set of exactly m vertices (it will eventually be from this family that we derive the recursive term in our lower bound). In particular, we will show that cut sets with deviations can be seen as equivalent to diagonal cutpaths. We will then deal with cut sets of more vertices with the Large Cut Set Lemma, showing through Lemmas 7 and 8 that a ranking corresponding to a cut set of more than m vertices will use no fewer colors than a particular ranking with a cut set of exactly m vertices. The recursive term in our lower bound then comes from computation of the largest square subgrid that exists in the families of induced subgrids from Lemma 6 ranging over all values of k .

Lemma 6. For $k \leq \lfloor \frac{m+2}{3} \rfloor$, any cut set of m vertices will induce subgraphs $G_{x,y}$, where

$$(x, y) \in \left\{ \left(m, \left\lceil \frac{m-k}{2} \right\rceil \right) \right\} \cup \left\{ (x', y') : x', y' > 0, x' = 2k - 3 - 2t, y' = \left\lceil \frac{m-k}{2} \right\rceil + t \right\}.$$

Let $S_{m,k}$ be the set of subgraphs $G_{x,y}$ induced by removing the cut set.

Proof. Because $|C(f)| = m$, each row in this path will contain exactly one vertex.

Pick two vertices v and v' of the cut set that occupy the leftmost column so that the cutpath of cut set vertices connecting them contains at least one vertex in the rightmost column and no other vertices in the leftmost column. The first condition has to be true or the cut set occupies fewer than k columns. The column sequence will be of the form $[0, c_1, c_2, \dots, c_i, c_{i+1} = k - 1, c_{i+2}, \dots, c_j = 0]$.

Consider the bottommost left-deviation $[c_l, c_{l+1}]$ and the topmost right-deviation $[c_r, c_{r+1}]$. Note that extending the lines of slope -1 and 1 containing c_{l+1} and c_{r+1} , respectively gives a diagonal cutpath falling within the boundaries of the cutpath, so the left connected component induced by removing a diagonal cutpath will be the subgraph of any cutpath with at least one deviation. An example of this argument is shown in Figure 9. Thus, for $|C(f)| = m$, we will only consider diagonal cutpaths.

Clearly, removing the cut set's k columns induces a $G_{\lceil \frac{m-k}{2} \rceil, m}$ subgraph. Consider the part of the induced subgraph in the same rows as the vertices in the left-diagonal cutpath. Then a rectangle with at least $\lceil \frac{m-k}{2} \rceil$ columns exists between the diagonal cutpath's two endpoints, and it occupies all rows between those of the cutpath's endpoints, the rectangle has a width of $2k - 3$ rows. Note that because a diagonal cutpath will occupy two vertices in each column of the grid except the last column, an extension of a grid $G_{x,y}$ that must exist will have dimensions $G_{x-2,y+1}$, giving us the desired set of rectangular subgrids that are guaranteed to exist for $G_{m,m}$. \square

Large Cut Set Lemma. For each ordered pair (x, y) so that there is an $x \times y$ subgrid in $S_{m,k}$ for a cut set of m vertices, if $|C(f)| > m$, then there is at least one subgrid $G_{x,y}$ of $G_{m,m}$ that contains at most $|C(f)| - m$ vertices.

To prove the Large Cut Set Lemma, we first prove two lemmas.

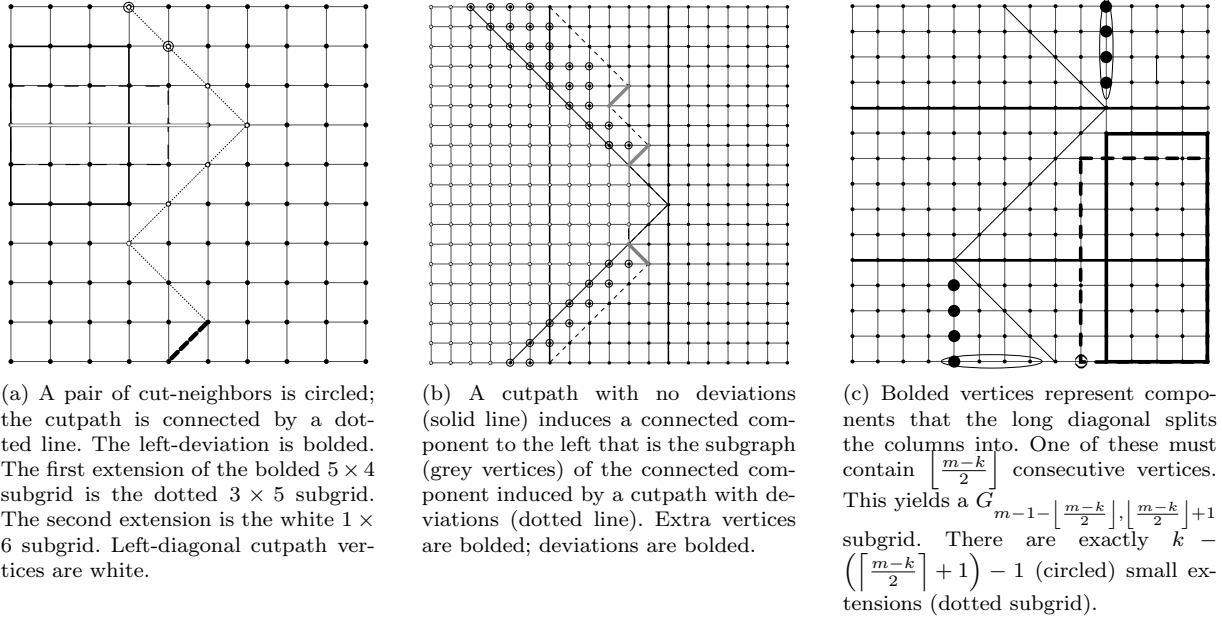


Figure 9: Examples of deviations and extensions and small extensions.

Lemma 7. *If the greatest lower bound on the rank number of all subgraphs induced by cut sets with exactly m vertices is r_{\min} , then the subgraph induced by cut set where at least two cut-neighbors are in the same column, has a rank number of at least r .*

Proof. As shown in Figure 10, there are nine configurations for a set of three consecutive columns where the middle column contains two cut-neighbors.

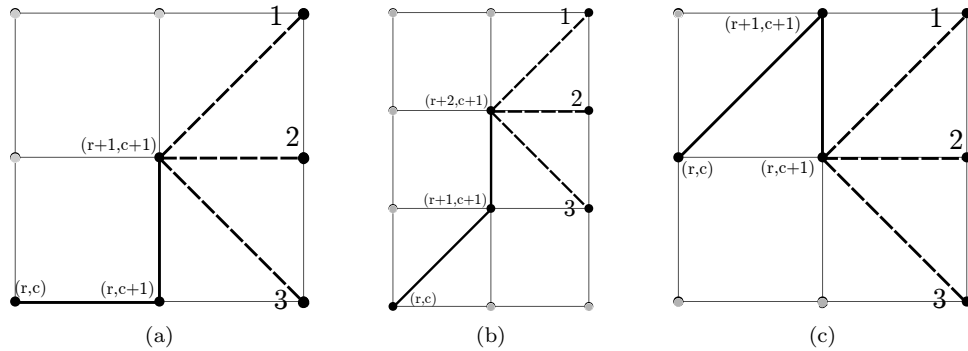


Figure 10: Cutpaths not obtainable by a set of deformations still satisfy the lower bound.

In the first three cases shown in Figure 10a, the sub-cutpath $[(r, c), (r, c + 1), (r + 1, c + 1)]$ is equivalent to a sub-cutpath of $[(r, c), (r + 1, c + 1)]$ with an extra cut vertex $(r, c + 1)$. If the induced subgraph of the cut set containing $[(r, c), (r + 1, c + 1)]$ has rank number of at least r_{\min} , adding $(r, c + 1)$ to the cut set will increase $|C(f)|$ by one while decreasing the rank number of the induced subgraph to the right of the cut set by at most 1, and it will still satisfy the lower bound.

Case 1 of Figure 10b is equivalent to the statement that cutpaths with deviations still satisfy the lower bound, proven in Lemma 6. Case 2 of Figure 10b follows by reasoning similar to the three cases in Figure 10a. In case 3 of Figure 10b, sub-cutpath $[(r, c), (r + 1, c + 1), (r + 2, c + 1), (r + 1, c + 2)]$ is equivalent to $[(r, c), (r +$

$1, c+1), (r+1, c+2)]$ with an extra vertex $(r+2, c+1)$, and if the induced subgraph of the cut set containing $[(r, c), (r+1, c+1), (r+1, c+2)]$ has rank number of at least r_{\min} , then adding $(r+2, c+1)$ will increase $|C(f)|$ by one while decreasing the rank number of the induced subgraph to the left of the cut set by at most 1

The three cases in Figure 10c follow by reasoning similar to that for case 3 of Figure 10b. \square

Thus, we only consider when cut-neighbors are in different columns. Observe that such cutpaths can be obtained through some sequence of deformations of and additions of vertices to a cutpath of length m . In particular, d deformations of vertices on a left-diagonal cutpath with endpoints $(0, 0)$ and $(0, 2k-2)$ will move the original endpoint to $(0, 2k-2+d)$. Then there are at least $|C(f)| + d$ vertices in the new cut set $C'(f)$. In general, if cut set $C'(f)$ contains a sub-cutpath that is obtained through a set of d deformations of $C(f)$, $C'(f) \geq C(f) + d$.

Lemma 8. *If the subgraph induced by a cut set, where all cut set vertices after $(0, 2k-1+d)$ in the cutpath are part of a diagonal cutpath, has a rank number of at least $r(\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1)$, then the subgraph induced by a cut set, where the vertices after $(0, 2k-1+d)$ are obtained by some deformations of a diagonal cutpath, has a rank number of at least $r(\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1)$.*

Proof. Consider a sub-cutpath beginning with $(0, 2k-1+d)$ and ending with the first vertex in the cutpath that lies on the line of slope 1 of vertices containing (k, k) , and let the subgraph induced by a cut set $C(f)$ that has this sub-cutpath contain $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. Let $C'(f)$ be the cut set with the same vertices as $C(f)$ before $(0, 2k-1+d)$ but with an additional deformation on the sub-cutpath after $(0, 2k-1+d)$. Then the extra deformation in $C'(f)$ will remove at most one more vertex from each subgrid in $S_{m,k}$ than will $C(f)$. However, $C'(f) \geq C(f) + 1$, so the rank number of the induced subgraph will stay the same or increase. \square

We now prove the Large Cut Set Lemma.

Proof. We prove by induction that the cut set of $|C(f)| > m$ vertices will occupy at most $|C(f)| - m = d$ vertices in at least one subgrid in $S_{m,k}$. Note that the extension of a subgrid will contain at most one more cut vertex than the subgrid because we are considering cut sets where columns contain no cut-neighbors in the same column. In columns after the $\lfloor \frac{d-1}{2} \rfloor$ th column, the diagonal cutpath beginning with $(0, 2k-1+d)$ occupies rows lower than a left-diagonal cutpath with endpoints $(0, 0)$ and $(0, 2k-1)$ would. Before this point, the j th extension contains at most two more cut set vertices than the $(j-1)$ th extension, so the $\lfloor \frac{d-1}{2} \rfloor$ th extension will have at most $1 + 2 \lfloor \frac{d-1}{2} \rfloor \leq d$ cut set vertices, and because the sub-cutpath beginning with $(0, 2k-1+d)$ is diagonal, it will have a vertex in the bottom row. Assume $G_{\lceil \frac{m-k}{2} \rceil, 2m-3-2d}$ contains at most d cut set vertices and a cut set vertex v in the bottom row. The extension of this will not contain v , but, again, it will contain at most one more cut set vertex than the subgrid, completing the induction. \square

Thus, we can assume that $|C(f)| = m$. We now prove our main result, that $r(m, m) \geq m + r(\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1)$.

Proof. We first show for each cut set column width, the subgrids in $S_{m,k}$ contain $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. Indeed, for $m = 5x+1, 5x+2, 5x+3, 5x+4$, and $5x+5$, $k = \lfloor \frac{m-1}{5} \rfloor + 2 = x+2$.

For $m = 5x+1$, the 0th subgrid extension has dimensions $(2x+1) \times (2x)$. $\lceil \frac{2m}{5} \rceil - 1 = 2x$, so the 0th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. For $m = 5x+2$, the 0th subgrid extension has dimensions $(2x+1) \times (2x)$. $\lceil \frac{2m}{5} \rceil - 1 = 2x$, so the 0th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. For $m = 5x+3$, the 0th subgrid extension has dimensions $(2x+1) \times (2x+1)$. $\lceil \frac{2m}{5} \rceil - 1 = 2x+1$, so the 0th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. For $m = 5x+4$, the 0th subgrid extension has dimensions $(2x+1) \times (2x+1)$. $\lceil \frac{2m}{5} \rceil - 1 = 2x+1$, so the 0th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. For $m = 5x+5$, the 0th subgrid extension has dimensions $(2x+1) \times (2x+2)$. $\lceil \frac{2m}{5} \rceil - 1 = 2x+1$, so the 0th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$.

Thus, for $k = \lfloor \frac{m-1}{5} \rfloor + 2$, the 0th subgrid extension must contain $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$ as a subgraph, so for $k \leq \lfloor \frac{m-1}{5} \rfloor + 2$, the 0th subgrid extension must contain this subgraph.

We now prove by induction that for $\lfloor \frac{m-1}{5} \rfloor + 2 < k \leq \lfloor \frac{m+2}{3} \rfloor$, one extension for each k must contain $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. If for i columns, the n th extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$, then the n th extension for $i + 1$ columns will have at least $\lceil \frac{2m}{5} \rceil - 1$ columns but exactly $\lceil \frac{2m}{5} \rceil + 1$ rows. The $(n + 1)$ th extension will have $\lceil \frac{2m}{5} \rceil$ rows and $\lceil \frac{2m}{5} \rceil$ columns.

However, note that for $k > \lfloor \frac{n+2}{3} \rfloor$, a diagonal cut set will not necessarily exist. If $|C(f)| = m$, then there will be a sub-cutpath with column sequence $[k - 1, k - 2, \dots, 1, 0]$, because a deviation will only increase the dimensions of the subgrids that are guaranteed to exist. Then either the left- or rightmost column of the cut set will contain at most $\lfloor \frac{m-k}{2} \rfloor$ consecutive vertices not belonging to the cut set. Of the remaining vertices on that column, only one is in the cut set, so the column will contain at least $m - 1 - \lfloor \frac{m-k}{2} \rfloor$ consecutive vertices not belonging to the cut set. This guarantees the existence of a $G_{m-1-\lfloor \frac{m-k}{2} \rfloor, \lfloor \frac{m-k}{2} \rfloor + 1}$ subgrid. Note that for $i < k - (\lfloor \frac{m-k}{2} \rfloor + 1)$, the $(i - 1)$ th extension of a subgrid will have exactly one more row than the i th extension. Call such an extension a *small extension*. For $i \geq k - (\lfloor \frac{m-k}{2} \rfloor + 1)$, the $(i - 1)$ th extension will have two more rows than the i th extension.

If $m - k$ is even, then after $k - (\lfloor \frac{m-k}{2} \rfloor + 1) - 1$ small extensions, we have a subgrid $G_{m-k+1, k-1}$. We prove by induction that if for i columns, the n th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$, where $i \leq k - (\lfloor \frac{m-k}{2} \rfloor + 1) - 1$, then for $i + 1$ columns, the $(n + 1)$ th subgrid extension contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$. Indeed, this is true for $k = \lfloor \frac{m+2}{3} \rfloor + 1$ columns. If this is true for i columns, then for $i + 1$ columns, the number of small extensions is at least one more than the number for i columns, so the $(n + 1)$ th extension is small and yields $G_{m-k+1, k-1}$, which contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$ by the inductive hypothesis.

If $m - k$ is odd, then after $k - (\lfloor \frac{m-k}{2} \rfloor + 1) - 1$ extensions, we have a subgrid $G_{m-k+2, k-2}$. We can prove this contains $G_{\lceil \frac{2m}{5} \rceil - 1, \lceil \frac{2m}{5} \rceil - 1}$ similarly. \square

Corollary 1. $r(m, m) \geq \frac{5}{3}m - \frac{25}{9}$.

Corollary 2. $\text{tri}_n \geq \frac{5}{3} \lfloor \frac{n}{2} \rfloor - \frac{34}{9}$.

Proof. This follows from the fact that tri_n contains $G_{\lfloor \frac{n}{2} \rfloor}$. \square

The rank number for triangle and square grids is now bounded between two linear functions, setting the groundwork for completely determining $r(m, m)$ and $\text{tri}(n)$.

6. Conclusion

In this paper, we have completely determined the rank number for $4 \times n$ grids, improved the upper bounds for the rank numbers of general grids, and improved the lower bounds for the rank numbers of square grids from a logarithmic to linear. As a corollary, we have improved the lower bounds for the rank numbers of triangle grids from logarithmic to linear as well.

Using our lower bound for square grid graphs, we may be able to determine the rank number function for general grid graphs. Firstly, we can generalize the Merging Lemma to all grid graphs: if $G_{m,n}$ with a sticky end has a λ -ranking and $G_{m,n-1}$ has a λ -ranking, then $G_{m,m+2n-1}$ has a $(\lambda + m)$ -ranking and $G_{m,3m+4n-2}$ has a $(\lambda + 2m)$ -ranking. Furthermore, the motivation behind the Corner Lemma may be extended to grids of higher dimensions. Finally, we conjecture that removing two corners in different columns from a sufficiently long grid will preserve the rank number of the grid. One avenue of further research is thus to prove or disprove this conjecture and to use this in conjunction with our lower bound and the generalization of the Merging Lemma in order to completely determine $r(m, n)$.

For sufficiently long grid graphs $G_{m,n}$, where $m \leq n$, we also conjecture that there always exist $r(m, n)$ -rankings where $|C(f)| = m$. Moreover, the new ideas about the cut set that we discuss in our proof for square grids are interesting objects of further study, in particular whether deformations or deviations will not yield rankings of fewer labels for other graphs.

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